OSCAR: A visionary, new computer algebra system

Wolfram Decker, William Hart,
Sebastian Gutsche, Michael Joswig

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Two Coordinated DFG-Programmes

Algorithmic and Experimental Methods

in Algebra, Geometry, and Number Theory

DFG Priority Project SPP 1489

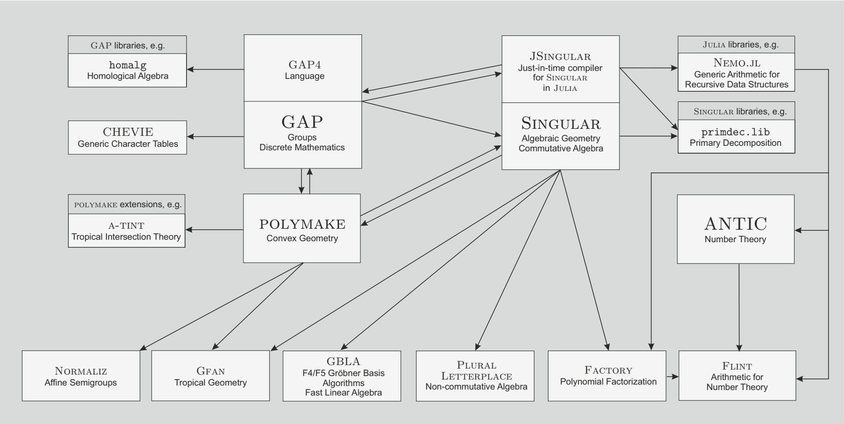
2010–2016

Collaborative Research Center Transregio TRR 195

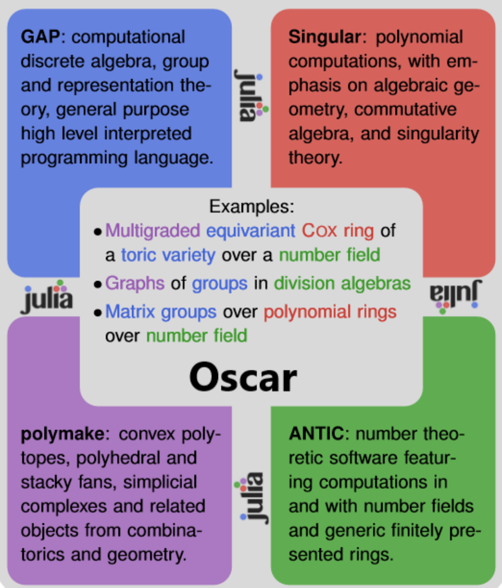
Symbolic Tools in Mathematics and their Application

2017–

Software Development Within SPP 1489



Software Development Within TRR 195



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Numerous small steps are needed to build OSCAR. Guiding principles:

- Take mathematical problems within TRR195 and international community into account.
- Most steps should be of immediate benefit for users (of current systems and OSCAR).
- Rely on existing resources where possible (e.g. Julia).

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Julia access to Singular KERNEL functions and data types:

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Author of Singular.jl:

- Bill Hart
- Oleksandr Motsak

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Primary Decomposition of Binomial Ideals

Some History

- David Eisenbud and Bernd Sturmfels: *Binomial Ideals*, 1996
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Example (Singular functions for binomial ideals)

Consider pure binomial ideal in three variables:

$$I = \langle x - y, x^3 - 1, zy^2 - z \rangle \subset \mathbb{C}[x, y, z].$$

Example for Immediate Benefits: Singular.jl
Primary Decomposition of Binomial Ideals

Example (Singular functions for binomial ideals)

```
julia  >  R,(x,y,z) = Singular.PolynomialRing(QabField(), ["x","y","z"])
julia  >  I = Ideal(R,x-y,x^3-1,z*y^2-z)
julia  >  isCellular(I)
(false,3)
julia  >  bcd=cellularDecomp(I)
2-element Array{Singular.sideal,1}:
julia  >  Singular.intersection(bcd[1], bcd[2])==I
true
julia  >  binomialPrimaryDecomposition(I)
3-element Array{Any,1}:
Singular Ideal over Singular Polynomial Ring (Coeffs(18)),(x,y,z),(dp(3),C)
with generators (y+(-1 in Q(z_1)), x+(-1 in Q(z_1)))
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- more expressive user language;
- a wealth of Julia features can be used

julia

- JIT compilation : near C performance.
- Designed by mathematically minded people.
- Open Source (MIT License).
- Actively developed since 2009.
- Supports Windows, OSX, Linux, BSD.
- Friendly C/Python-like (imperative) syntax.

Nemo.jl:

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AbstractAlgebra.jl:

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GAP.jl

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- Interface with Gap: ability to call Julia functions from Gap and vice versa

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- Class and unit group computation
- Pseudo-Hermite normal form for modules over Dedekind domains
- Beginnings of class field theory and relative extensions

Oscar.jl (future work)

- Projects that make use of all of the above
- Present a consistent view of mathematics to the user: no need to worry about which implementation of the integers is being used behind the scenes
- Explore how far the Julia language can be pushed for computer algebra

Example improvement: Minpoly over $\mathbb{Z}$

Theorem

Suppose M is a linear operator on a K-vector space V, and that $V = W_1 + W_2 + \cdots + W_n$ for invariant subspaces W_i. Then the minimal polynomial of M is $LCM(m_1, m_2, \ldots, m_n)$, where m_i is the minimal polynomial of M restricted to W_i.

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The subspaces we have in mind are the following:

Definition

Given a vector v in a vector space V the *Krylov subspace* $K(V, v)$ associated to v is the linear subspace spanned by $\{v, Mv, M^2v, \ldots\}$.

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Idea:

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Too expensive to evaluate the minpoly $m(T)$ at M. Need a better termination condition.

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- Can be checked using Matrix-Vector products, which are cheap
- Leads to worst case $O(n^4)$ algorithm, but generically $O(n^3)$

Characteristic and minimal polynomial

Table: Charpoly and minpoly timings

Op	Sage 6.9	Pari 2.7.4	Magma 2.21-4	Giac 1.2.2	Flint
Charpoly	0.2s	0.6s	0.06s	0.06s	0.04s
Minpoly	0.07s	>160 hrs	0.05s	0.06s	0.04s

for 80×80 matrix over $\mathbb{Z}$ with entries in $[-20, 20]$ and minpoly of degree 40.

Minimal polynomial over $\mathbb{Z}[x]$

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- Possibility to add compiled Julia functions as kernel functions to GAP

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- Nested lists of the above to Arrays

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```
function orbit( self, element, generators, action )
  work_set = [ element ]
  return_set = [ element ]
  generator_length = gap_LengthPlist(generators)
  while length(work_set) != 0
    current_element = pop!(work_set)
    for current_generator_number = 1:generator_length
      current_generator = gap_ListElement(generators,
                                          current_generator_number)
      current_result = gap_CallFunc2Args(action,current_element,
                                         current_generator)
      is_in_set = false
      for  i in return_set
        if i == current_result
          is_in_set = true
          break
        end
      end
      if ! is_in_set
        push!( work_set, current_result )
        push!( return_set, current_result )
      end
    end
  end
  return return_set
end
```

JuliaInterface: Julia functions as kernel modules

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gap> orbit_c( 1, S, OnPoints );; time;
46
```

From the GAP side

How does GAP benefit from OSCAR (except mathematical algorithms)?

From the GAP side

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Speedup

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Benefits:

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- Julia has more functionality available in its standard library than C
- Julia may be easier to use then C

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Language features

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- Flexible type system: Objects can learn about themselves
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- Immediate propagation: Second execution layer is used to spread properties between objects
- Categorical programming language as defined in the CAP project

Category theory as programming language

Category theory

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CAP - Categories, Algorithms, Programming

CAP implements a
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(j/w Sebastian Posur)

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A B C

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$$
\begin{tikzcd}
A \arrow[r, red] & B & C
\end{tikzcd}
$$

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Some categorical operations in abelian categories

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- Zero morphisms

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Implementation of the kernel

Let $\varphi \in \mathrm{Hom}(A, B)$.

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... one needs an object $\ker \varphi$,

$$\begin{array}{ccc} \ker \varphi & & \\ & A \xrightarrow{\varphi} B \end{array}$$

Implementation of the kernel

Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

... one needs an object $\ker \varphi$,
its embedding $\kappa = \mathrm{KernelEmbedding}(\varphi)$,

$$\begin{tikzcd} \ker \varphi \arrow[dr, hook, "\kappa"] & & \\ & A \arrow[r, "\varphi"] & B \end{tikzcd}$$

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Let $\varphi \in \mathrm{Hom}(A, B)$. To fully describe the kernel of φ ...

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and for every test morphism τ

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
& A \arrow[r, "\varphi"] & B \\
T \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

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a *unique* morphism $\lambda = \mathrm{KernelLift}(\varphi, \tau)$

$$\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] & & \\
 & A \arrow[r, "\varphi"] & B \\
T \arrow[uu, dashed, "\lambda"] \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
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a *unique* morphism $\lambda = \mathrm{KernelLift}(\varphi, \tau)$, such that

$$
\begin{tikzcd}
\ker\varphi \arrow[dr, hook, "\kappa"] \arrow[drr, bend left, "0"] \arrow[d, phantom, "\circlearrowright", shift left=4ex] & & \\
 & A \arrow[r, "\varphi"] & B \\
T \arrow[uu, dashed, "\lambda"] \arrow[ur, "\tau"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

What is CAP ?

CAP - Categories, Algorithms, and Programming

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CAP is a framework to implement computable categories and provides

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CAP is a framework to implement computable categories and provides

- specifications of categorical operations
- generic algorithms based on basic categorical operations
- a categorical programming language having categorical operations as syntax elements

Computing the intersection

Let $M_1 \subseteq N$ and $M_2 \subseteq N$ subobjects.

Computing the intersection

Let $M_1 \hookrightarrow N$ and $M_2 \hookrightarrow N$ subobjects.

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$$
\begin{tikzcd}
M_1 \arrow[dr, hook, "\iota_1"] & \\
& N \\
M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}
$$

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$$\begin{tikzcd}
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- $\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right),\ i = 1, 2$

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 & & M_1 \arrow[dr, hook, "\iota_1"] & \\
M_1 \cap M_2 \arrow[r, hook, "\kappa"] & M_1 \oplus M_2 \arrow[ur, two heads, "\pi_1"] \arrow[rr, "\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2"] \arrow[dr, two heads, "\pi_2"'] & & N \\
 & & M_2 \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

- $\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left(\left(M_1, M_2\right), i\right),\ i = 1, 2$
- $\varphi := \iota_1 \circ \pi_1 - \iota_2 \circ \pi_2$
- $\kappa := \mathrm{KernelEmbedding}\left(\varphi\right)$
- $\gamma := \iota_1 \circ \pi_1 \circ \kappa$

Translation to CAP

$\pi_i := \mathrm{ProjectionInFactorOfDirectSum}\left((M_1, M_2), i\right),\ i = 1, 2$

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pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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```
lambda := PostCompose( iota1, pi1 );
phi := lambda - PostCompose( iota2, pi2 );
```

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$\gamma := \iota_1 \circ \pi_1 \circ \kappa$

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```

$\kappa := \mathrm{KernelEmbedding}\left(\varphi\right)$

```
kappa := KernelEmbedding( phi );
```

$\gamma := \iota_1 \circ \pi_1 \circ \kappa$

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```

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```
kappa := KernelEmbedding( phi );
```

$\gamma := \iota_1 \circ \pi_1 \circ \kappa$

```
gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
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```

```
lambda := PostCompose( iota1, pi1 );
phi := lambda - PostCompose( iota2, pi2 );
```

```
kappa := KernelEmbedding( phi );
```

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gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
lambda := PostCompose( iota1, pi1 );
phi := lambda - PostCompose( iota2, pi2 );
kappa := KernelEmbedding( phi );
gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
IntersectionOfObjects := function( iota1, iota2 )

  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
IntersectionOfObjects := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );

  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );

  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );

  kappa := KernelEmbedding( phi );

  gamma := PostCompose( lambda, kappa );
```

Translation to CAP

```
IntersectionOfObjects := function( iota1, iota2 )

  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Translation to CAP

```
IntersectionOfObjects := function( iota1, iota2 )
  local M1, M2, pi1, pi2, lambda, phi, kappa, gamma;
  M1 := Source( iota1 );
  M2 := Source( iota2 );
  pi1 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 1 );
  pi2 := ProjectionInFactorOfDirectSum( [ M1, M2 ], 2 );
  lambda := PostCompose( iota1, pi1 );
  phi := lambda - PostCompose( iota2, pi2 );
  kappa := KernelEmbedding( phi );
  gamma := PostCompose( lambda, kappa );
  return gamma;
end;
```

Mathematica: Integration

```
In[1]:= Integrate[ArcTan[x] - ArcCot[1/x], {x,0,1}]

Out[1]= 0

In[2]:= Integrate[ArcTan[x] - ArcCot[1/x], {x,0,1.0}]

                        -15
Out[2]= -7.88258 10
```

Mathematica: Integration

```
In[1]:= Integrate[ArcTan[x] - ArcCot[1/x], {x,0,1}]

Out[1]= 0

In[2]:= Integrate[ArcTan[x] - ArcCot[1/x], {x,0,1.0}]

                          -15
Out[2]= -7.88258 10

In[3]:= FullSimplify[ArcTan[x] - ArcCot[1/x]]

Out[3]= 0
```

Sage: Gröbner Bases

```
a = 1
b = 2
c = -3

x,y = QQ['x,y'].gens()
f = a*x^3*y^2+b*x+y^2+1
g = c*x*y^4+x^3+y
I = ideal(f,g)
B = I.groebner_basis(); B

[y^6 + 1/3*x^2*y^3 - 1/3*x^2*y^2 + y^4
     - 1/3*x^2 + 2/3*y,
 x^5 + 3*y^4 + x^2*y + 6*x*y^2 + 3*y^2,
 x^3*y^2 + y^2 + 2*x + 1,
 x*y^4 - 1/3*x^3 - 1/3*y]
```

Sage: Plotting Curves

```
var('x,y')
f = a*x^3*y^2+b*x+y^2+1
g = c*x*y^4+x^3+y

C = implicit_plot(f, (x,-2,2), (y,-2,2),
    cmap=['red'], plot_points=150, fill=False)
D = implicit_plot(g, (x,-2,2), (y,-2,2),
    cmap=['blue'], plot_points=150, fill=False)
C+D
```